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On source and vortex distributions in the linearised theory of steady supersonic flow

-by-

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-SUMMARY-

The hyperbolic character of the differential equation satisfied by the velocity potential in linearised supersonic flow entails the presence of fractional infinities in the fundamental solutions of the equation. Difficulties arising from this fact can be overcome by the introduction of Hadamard's 'finite part of an infinite integral'. Together with the definition of certain counterparts of the familiar vector operators this leads to a natural development of the analogy between incompressible flow and linearised supersonic flow. In particular, formulae are derived for the field of flow due to an arbitrary distribution of supersonic sources and vortices.

Applications to Aerofoil theory, including the calculation of the downwash in the wake of an aerofoil, are given in a separate report.

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1. Introduction

It is well known that the elementary solution of Laplace's equation in three dimensions - i.e., the velocity potential of a source in Hydrodynamics, and the potential of a gravitating particle in Newtonian potential theory - has a counterpart in the linearised theory of supersonic flow, viz., the velocity potential of the so-called 'supersonic source'. However, the development of the analogy meets with obstacles which are largely due to the fact that the velocity potential of a supersonic source becomes infinite not only at the actual origin of the source but also everywhere on the Mach cone emanating from it. Thus, in trying to evaluate the flow across a surface surrounding a supersonic source, the resultant integral becomes infinite. This and other difficulties can be overcome by the introduction of the concept of the finite part of an infinite integral, which was first defined by Hadamard (ref.1) in connection with the solution of initial value problems for hyperbolic partial differential equations. A description of this concept is given in para.2 below, and applications to problems concerning source and doublet distributions under steady supersonic conditions will be found in paras. 3 and 4.

The concept of a vortex in the linearised theory of supersonic flow was first considered by Schlichting (ref.2) who obtained the field of flow corresponding to a 'horseshoe vortex' by a synthesis of doublets. Schlichting's approach has been the subject of some criticism as in certain respects the supersonic horseshoe-vortex is different out of all recognition from its subsonic counterpart. However, it is shown in para.5 below that more generally the field of flow due to an arbitrary vorticity distribution, under steady supersonic conditions can be calculated in strict analogy with the method due to Stokes and Helmholtz in classical Hydrodynamics. The results are in agreement with Schlichting's for the particular case of a horseshoe vortex.

Applications to Aerofoil theory will be given in a separate report.

2. The Finite Part of an Infinite Integral

Let $D(x,y,z)$ be an algebraic function of three variables so that the equation $D(x,y,z) = 0$ determines a surface Σ in three-dimensional space. The surface Σ divides space into disconnected components Σ_n in which $D(x,y,z)$ is of constant sign; also, $D(x,y,z)$ will be supposed to change sign across any ordinary point of Σ. Further let $f(x,y,z)$ be a real function defined in a certain region R such that

$$f(x,y,z) = g(x,y,z) + g_0(x,y,z)\,D^{-n/2} + g_1(x,y,z)\,D^{-n/2+1} \ldots + \ldots\ldots$$
$$+ g_k(x,y,z)D^{-\frac{1}{2}} + g_{k+1}(x,y,z)D^{+\frac{1}{2}} + \ldots + g_{k+m}(x,y,z)D^{m-\frac{1}{2}} \ldots \qquad \ldots\ldots(1)$$

/where....

where n is a positive odd integer, $n = 2k+1$, $k = 0,1,2,\ldots$, and the functions $g(x,y,z)$, $g_0(x,y,z)$, ... are either all analytic everywhere except on Σ, or at least have derivatives of a sufficiently high order, which are bounded in the neighbourhood of Σ. At any rate it is assumed that the analytic expressions for these functions may be different for the different Σ_n.

Given a small positive quantity ϵ, we denote by $N(\epsilon)$ the set of all points s which satisfy an inequality $|s - s_0| \leq \epsilon$ for at least one point s_0 of Σ. We denote the boundary of $N(\epsilon)$ by $B(\epsilon)$, and we denote by $R(\epsilon)$ the region obtained by excluding from R all the points of $N(\epsilon)$. Furthermore, given a curve C, a surface S, or a volume V in R, we denote by $C(\epsilon)$, $S(\epsilon)$, and $V(\epsilon)$ the subsets of C, S, and V respectively which are obtained by the exclusion of the points of $N(\epsilon)$.

The concept of the finite part $^*J$ of an (finite or infinite) integral J - where J is any line, surface, or volume integral of f on C, S, or V respectively, e.g.

$$\int_C f dx, \int_S f dxdy, \int_V f dxdydz,$$

where C, S, and V are supposed to be bounded - will now be defined as follows.

Given the formal expression for J, we denote by $J(\epsilon)$ the corresponding integral taken over $C(\epsilon)$, $S(\epsilon)$, or $V(\epsilon)$ only. Subject to the specified conditions of regularity, $J(\epsilon)$ will be finite and of the form

$$J(\epsilon) = a_0 \epsilon^{-n/2+1} + \ldots + a_{k-1} \epsilon^{-\frac{1}{2}} + O(1) \ldots (2)$$

where $O(1)$, as usual, denotes a function which remains finite as ϵ tends to 0. We then define $^*J$ by

$$^*J = \lim_{\epsilon \to 0} \left(J(\epsilon) - a_0 \epsilon^{-n/2+1} - \ldots - a_{k-1} \epsilon^{-\frac{1}{2}} \right) \ldots (3).$$

As stated in the introduction, the concept of the finite part of an infinite integral is due to Hadamard (ref.1), whose definition, however, applies to a more restricted type of integral only. Hadamard writes $\overline{\int} J$ instead of our $^*J$ which is used by Courant and Hilbert (ref.3).

It will be seen that if J is finite then $^*J = J$. Also, the finite part of an (infinite) integral is invarient with respect to a transformation of coordinates, provided the Jacobian of the transformation does not vanish on Σ. In particular, if we are dealing with the finite parts of integrals involving vector quantities, the result is independent of a rotation of coordinates.

/There....

There will be no occasion for confusion if in future we refer to the finite part of a (finite or infinite) integral simply as 'a finite part'.

The finite parts of m-fold integrals in n-dimensional space, $n > 3$, $m \leq n$ can be defined in a strictly analogous manner. The rules valid for them are, mutatis mutandis, the same as for finite parts in three dimensions.

The rules of calculation with finite parts, such as the rules of addition, are the same as for ordinary integrals. Also, if f depends on a parameter λ, but D is fixed, then - provided the g functions are sufficiently regular (e.g., if they are analytic in the various Σ_n - it is not difficult to show that we may differentiate under the sign of the integral, e.g.,

$$\frac{d}{d\lambda}\left(\overset{*}{\int_C} f dx\right) = \overset{*}{\int_C} \frac{\partial f}{\partial \lambda} dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Under similar conditions the finite part of a multiple integral may be obtained by successive integration (including the operation of taking the finite part) with respect to the independent variables involved, taken in any arbitrary order. Thus, with the appropriate limits we have, for instance,

$$\overset{*}{\int} f dx dy dz = \overset{*}{\int}\left(\overset{*}{\int}\left(\overset{*}{\int} f dx\right) dy\right) dz \quad \ldots\ldots\ldots (5)$$

More generally, we shall encounter cases where D, and therefore Σ, depends algebraically on one or more parameters. We are going to show (i) that even in that case we may 'differentiate under the integral sign' and (ii) that if a given integral, or finite part, involves integration with respect to such parameters, as well as with respect to one or more of the space coordinates, we may exchange the order of integration without affecting the value of the integral.

To see this, we increase the ordinary three dimensions of space x, y, z, by the parameter or parameters involved. Then in the augmented space, the surface $D = 0$ is again fixed, and in order to prove our assertions, it is sufficient to show (i)′ that in order to find the derivative of a finite part in n-dimensional space with respect to any variable which is not involved in the integration, we may differentiate under the sign of the integral, and (ii)′ that for any multiple integral in n-dimensional space, $1 < m \leq n$, we have

$$\overset{*}{\int}\left(\overset{*}{\int} dx_1\right) dx_2 \ldots dx_m = \overset{*}{\int} f dx_1\, dx_2 \ldots dx_m$$

taken over the appropriate regions. It is clear that (ii)′ would prove (ii), by induction.

/We......

We may reduce (i)′ to (ii)′. In fact, (i)′ states explicitly that

$$\frac{\partial}{\partial x_m} \overset{*}{\int} f dx_1 \ldots dx_{m-1} = \overset{*}{\int} \frac{\partial f}{\partial x_m} dx_1 \ldots dx_{m-1} ,$$

and this will be proved if it can be shown that

$$\overset{*}{\int} f dx_1 \ldots dx_{m-1} = \overset{*}{\int}\left(\overset{*}{\int} \frac{\partial f}{\partial x_m} dx_1 \ldots dx_{m-1}\right) dx_m + C,$$

where the lower limit of the integral with respect to x_m is arbitrary and C is independent of x_m.
Putting

$$F = \frac{\partial f}{\partial x_m} ,$$

we have

$$f = f_o + \overset{*}{\int} F dx_m ,$$

where f_o is the value of f for an arbitrary but definite value of x_m (for given $x_1, \ldots, x_{m-1}$), and the integral is taken with that particular value of x_m as lower limit.
Now, assuming that (ii)′ has been proved we have

$$\overset{*}{\int}\left(\overset{*}{\int} F dx_m\right) dx_1 \ldots dx_{m-1} = \overset{*}{\int}\left(\overset{*}{\int} F dx_1 \ldots dx_{m-1}\right) dx_m$$

and so

$$\overset{*}{\int}\left(f - f_o\right) dx_1 \ldots dx_m = \overset{*}{\int}\left(\overset{*}{\int} \frac{\partial f}{\partial x_m} dx_1 \ldots dx_{m-1}\right) dx_m,$$

i.e.,

$$\overset{*}{\int} f\, dx_1 \ldots dx_{m-1} = \overset{*}{\int}\left(\overset{*}{\int} \frac{\partial f}{\partial x_m} dx_1 \ldots dx_{m-1}\right) dx_m + \overset{*}{\int} f_o dx_1 \ldots dx_m,$$

and the last term is independent of x_m, as required.

To establish (ii)′, we have to prove

$$\overset{*}{\int}\left(\overset{*}{\int} f\, dx_m\right) dx_1 \ldots dx_{m-1} = \overset{*}{\int} f\, dx_1 \ldots dx_m \quad \ldots (6)$$

Putting $\overset{*}{\int} f\, dx_m = F$, we see that (6) becomes

$$\overset{*}{\int}_S F\, dx_m \ldots dx_{m-1} = \overset{*}{\int}_R \frac{\partial F}{\partial x_m} dx_1 \ldots dx_m \quad \ldots\ldots (7)$$

/taken....

taken over a certain region R on the right hand side and over its boundary S on the left hand side respectively. This is essentially the theorem of Gauss (or Green) for higher spaces. For $m \leq 3$, this theorem will be proved below for finite parts (without relying on the results of the present discussion), and the proof for greater m is quite similar.

An important example of a finite part will now be calculated. Let $D(x,y,z)$ be defined by $D \equiv x^2 - \beta^2(y^2+z^2)$ and $f(x,y,z)$ by

$$\begin{cases} f(x,y,z) = \dfrac{\sigma x}{\left[x^2-\beta^2\left(y^2+z^2\right)\right]^{3/2}} \quad \text{for } x^2 > \beta^2(y^2+z^2),\ x>0 \\ \text{and} \\ f(x,y,z) = 0 \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

elsewhere where σ and β are arbitrary constants. We find that all the conditions laid down at tge beginning of this paragraph are satisfied in every region R not including the origin, the surface Σ being given by the cone $x^2 - \beta^2(y^2+z^2) = 0$.

Further, let the open surface S be given by $x = \alpha,\ y^2+z^2 \leq r^2$ where $\alpha > 0$ and $r > \frac{\alpha}{\beta}$. S is a circular area including the circle $x = \alpha$, $y^2+z^2 = \frac{\alpha^2}{\beta^2}$, on which f becomes infinite of order 3/2. We are going to evaluate $^*J = {}^*\!\!\int_S f\,dy\,dz$

Given $\epsilon > 0$, let $S_1(\epsilon)$ be the points of $S(\epsilon)$ for which $a^2 > \beta^2(y^2+z^2)$, and $S_2(\epsilon)$ the complementary set of $S(\epsilon)$. Then f vanishes on $S_2(\epsilon)$ and so

$$J(\epsilon) = \int_{S(\epsilon)} f\,dx\,dy = \int_{S_1(\epsilon)} f\,dx\,dy + \int_{S_2(\epsilon)} f\,dx\,dy = \int_{S_1(\epsilon)} f\,dx\,dy =$$

$$= \int_{S_1(\epsilon)} \frac{\sigma\alpha\,dx\,dy}{\left[\alpha^2-\beta^2\left(y^2+z^2\right)\right]^{3/2}} = \frac{\sigma}{\beta^2}\int_0^{2\pi}\int_0^{\frac{\beta\eta}{\alpha}} \frac{\rho\,d\rho\,d\theta}{\left(1-\rho^2\right)^{3/2}}$$

where $y = \frac{\alpha}{\beta}\rho\cos\theta$, $z = \frac{\alpha}{\beta}\rho\sin\theta$, and η is the radius of the circle bounding $S_1(\epsilon)$. It is easy to deduce from the definition of $S(\epsilon)$ that (compare Fig.1) $\eta = \frac{1}{\beta}\left(\alpha - \sqrt{1+\beta^2}\right)$.

We then obtain

$$J(\epsilon) = \frac{2\pi\sigma}{\beta^2}\left[\frac{1}{(1-\rho^2)^{1/2}}\right]_0^{\frac{\beta}{\alpha}\eta} = \frac{2\pi\sigma}{\beta^2}$$

$$\left\{\frac{1}{\left[\left(2-\epsilon\frac{\sqrt{1+\beta^2}}{\alpha}\right)\epsilon\,\frac{\sqrt{1+\beta^2}}{\alpha}\right]^{1/2}} - 1\right\}$$

$$= \frac{2\pi\sigma}{\beta^2}\left\{\alpha^{\frac{1}{2}}\left(2\sqrt{1+\beta^2}\right)^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}} - 1 + O\left(\epsilon^{\frac{1}{2}}\right)\right\}$$

/and.....

and so

$$ {}^{*}J = {}^{*}\!\!\int_S \frac{\sigma\, x\, dy\, dz}{\left[x^2 - \beta^2\left(y^2 + z^2\right)\right]^{3/2}} = -\frac{2\pi\sigma}{\beta^2} \quad \text{.....(8)} $$

This result is of fundamental importance for subsequent developments.

We shall also require extensions of the divergence theorem of Gauss and Green and of the curl theorem of Stokes to finite parts. Particular cases of the divergence theorem are in fact proved and applied by Hadamard in the above mentioned book.

A function f will be called an admissible function if it satisfies the conditions laid down at the beginning of this paragraph. A vector function f will be said to be admissible if its components are admissible - and this is independent of the system of coordinates. If all the first derivatives of the components of f are admissible, then div f also is admissible. We are going to show that under these conditions we have for any volume V bounded by a surface S such as considered in the ordinary divergence theorem,

$$ {}^{*}\!\!\int_S \underline{f}\, d\underline{s} = {}^{*}\!\!\int_V \operatorname{div} \underline{f}\, dV \quad \text{.................(9)} $$

By equation (1), the vector $\underline{f} = (f^1, f^2, f^3)$ can be divided into two parts, $\underline{F} = (F^1, F^2, F^3)$, and $G = (G^1, G^2, G^3)$, $\underline{f} = \underline{F}\ \underline{G}$, so that the F^i are finite and continuous on Σ while the G^i become infinite there. Then div $\underline{F}$ is either finite and continuous everywhere in its domain of definitions or it becomes infinite of order $\frac{1}{2}$ on Σ. Even then $\int_V \operatorname{div} \underline{F}\, dV$ exists as an ordinary improper integral, and $\int_S \underline{F} d\underline{S}$ $\int_V \operatorname{div} \underline{F}\, dV$. Since div $\underline{f}$ = div $\underline{F}$ + div $\underline{G}$, it is therefore sufficient to show that

$$ {}^{*}\!\!\int_S \underline{G}\, d\underline{S} = {}^{*}\!\!\int_V \operatorname{div} \underline{G}\, dV $$

i.e.,

$$ {}^{*}\!\!\int_S \underline{G}\, d\underline{s} - {}^{*}\!\!\int_V \operatorname{div} \underline{G}\, dV = 0 \quad \text{............(10)} $$

Let $S'(\epsilon)$ be the product of (the set of points common to) V and $B(\epsilon)$. Then $V(\epsilon)$ is bounded by $S(\epsilon) + S'(\epsilon)$, the sumset of $S(\epsilon)$ and $S'(\epsilon)$. Hence, applying the divergence theorem to the volume $V(\epsilon)$, we obtain

$$ \int_{S(\epsilon)} \underline{G}\, d\underline{S} + \int_{S'(\epsilon)} \underline{G}\, d\underline{S} = \int_{V(\epsilon)} \operatorname{div} \underline{G}\, dV $$

/or.........

or. $$\int_{S(\epsilon)} \underline{G}\, d\underline{S} - \int_{V(\epsilon)} \operatorname{div} \underline{G}\, dV = -\int_{S'(\epsilon)} \underline{G}\, d\underline{S} \quad \ldots(11)$$

Now $\int_{S(\epsilon)} \underline{G}\, d\underline{S}$ is of the form

$$\int_{S(\epsilon)} \underline{G}\, d\underline{S} = a_o \epsilon^{-(n/2 - 1)} + \cdots + a_{k-1} \epsilon^{-\frac{1}{2}} + {}^{*}\!\!\int_S \underline{G}\, d\underline{S} + H(\epsilon)$$

where $H(\epsilon)$ is a function which tends to 0 as ϵ tends to 0, and similarly $\int_{V(\epsilon)} \operatorname{div} \underline{G}\, d\underline{s}$ will be seen to be of the form

$$\int_{V(\epsilon)} \operatorname{div} \underline{G}\, dV = b_o \epsilon^{-n/2} + \cdots + b_k \epsilon^{-\frac{1}{2}} + {}^{*}\!\!\int_V \operatorname{div} \underline{G}\, dV + K(\epsilon)$$

where $K(\epsilon)$ is a function which tends to 0 as ϵ tends to 0.

In other words $\int_{S(\epsilon)} \underline{G}\, d\underline{S}$ and $\int_{V(\epsilon)} \operatorname{div} \underline{G}\, dV$ differ from ${}^{*}\!\!\int_S \underline{G}\, d\underline{S}$ and ${}^{*}\!\!\int_V \operatorname{div} G\, dV$ respectively only by vanishing functions of ϵ and by fractional infinities of ϵ. Hence, in order to prove (10), it is, by (11), sufficient to show that

$$\int_{S'(\epsilon)} \underline{G}\, d\underline{S} = c_o \epsilon^{-n/2} + \cdots + c_k \epsilon^{-\frac{1}{2}} + L(\epsilon) \quad ..(12)$$

where $L(\epsilon)$ is a function which tends to 0 as ϵ tends to 0. And (12) can be readily deduced from the fact that the components of $\underline{F}$ satisfy conditions of the type indicated by (1). In fact (1) shows that on $S'(\epsilon)$ G^1 is of the type

$$G^1 = C_o \epsilon^{-n/2} + \cdots + C_k \epsilon^{-\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

where the C depend on the parameters of $S'(\epsilon)$, and similar expressions hold for the other components of $\underline{G}$.

Next, let $\underline{f}$ be a vector function of the same description as before, and let J be an open surface bounded by a curve C such as considered in the ordinary curl theorem. Under these conditions we are going to show that

$${}^{*}\!\!\int_C \underline{f}\, d\underline{\ell} = {}^{*}\!\!\int_S \operatorname{curl} \underline{f}\, d\underline{S} \quad \ldots\ldots\ldots\ldots\ldots(14)$$

Splitting $\underline{f}$ into two parts $\underline{F}$ and $\underline{G}$ as before, we first show that

$${}^{*}\!\!\int_C \underline{G}\, d\underline{\ell} = {}^{*}\!\!\int_S \operatorname{curl} \underline{G}\, d\underline{S} \quad \ldots\ldots\ldots\ldots\ldots(15).$$

/Let.....

Let $C'(\epsilon)$ be the product set of S and $B(\epsilon)$. Then $S(\epsilon)$ is bounded by $C(\epsilon) + C'(\epsilon)$ and so, applying Stokes' theorem to $S(\epsilon)$, we obtain

$$\int_{C(\epsilon)} \underline{G}\, d\underline{\ell} + \int_{C'(\epsilon)} \underline{G}\, d\underline{\ell} = \int_{S(\epsilon)} \text{curl } \underline{G}\, d\underline{S} \text{ ....(16).}$$

In order to be able to deduce (15) from (16) we have to show, similarly as in the proof of the divergence theorem that

$$\int_{C'(\epsilon)} \underline{G}\, d\underline{\ell} = c_0 \epsilon^{-n/2} + \dots + c_k \epsilon^{-\frac{1}{2}} + H(\epsilon) \text{ ..(17)}$$

where $\lim_{\epsilon \to 0} L(\epsilon) = 0$, and this follows from (1), as before.

We still have to prove that

$$\overset{*}{\int_C} \underline{F}\, d\underline{\ell} = \overset{*}{\int_S} \text{curl } \underline{F}\, d\underline{S} \quad \text{..................(18)}$$

This is obvious, by Stokes' Theorem, if curl $\underline{F}$ remains finite everywhere, and if curl $\underline{F}$ becomes infinite on Σ (in which case the right hand side of (18) is an ordinary improper integral), provided S has not got a finite area in common with Σ. Assuming on the contrary that S has a two dimensional subset S bounded by $\overline{C}$ in common with Σ, it is then sufficient to show that

$$\overset{*}{\int_{\overline{C}}} \underline{F}\, d\underline{\ell} = \overset{*}{\int_{\overline{S}}} \text{curl } \underline{F}\, d\underline{S} \text{ ....(19)}$$

Again, since curl $\underline{F}$ is an admissible vector, it follows that $\underline{F}$ is of the form $\underline{F} = \underline{F}_1 + \underline{F}_2$ where the components of $\underline{F}_1$ are finite and continuous on Σ and the components of $\underline{F}_2$ vanish on Σ (so that $\underline{F}_2 | \epsilon^{-\frac{1}{2}}$ remains bounded as ϵ tends to 0.)
Hence

$$\overset{*}{\int_{\overline{C}}} \underline{F}\, d\underline{\ell} = \int_{\overline{C}} \underline{F}_1\, d\underline{\ell}\,.$$

On the other hand by the definition of the finite part,

$$\overset{*}{\int_{\overline{S}}} \text{curl } \underline{F}\, d\underline{S} = \overset{*}{\int_{\overline{S}}} \text{curl } \underline{F}_1\, d\underline{S},$$ and by Stokes' theorem

/for......

for finite functions $\int_{\bar{C}} \underline{F}_1 \, d\underline{\ell} = \int_{\bar{S}} \underline{\text{curl}}\, F_1 \, d\underline{S}$

and so $\displaystyle \int^{*}_{\bar{C}} \underline{F}\, d\underline{\ell} = \int^{*}_{\bar{S}} \text{curl}\, \underline{F}\, d\underline{S}$, as required.

Equations (19) is now established completely.

3. First applications to the linearised theory of steady supersonic flow

In this and the following section we are going to discuss solutions of the equation

$$-\beta^2 \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \ldots\ldots\ldots\ldots(20)$$

in relation to the linearised theory of supersonic flow. The following details are not intended as an exhaustive introduction to that theory; their purpose merely is to establish and explain the terminology used in the sequel.

Assume that the free stream velocity of the given field of flow is parallel to the positive direction of the x-axis ahd is of magnitude U where U is greater than the speed of sound $a = \sqrt{\frac{dp}{d\rho}}$.

Calling the total velocity components in the direction of the x, y, and z-axes, u, v, and w respectively, and assuming that u is large compared with v and w, and compared with its difference from U, we obtain, for steady conditions, the linearised Eulerian equations

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = U \frac{\partial u}{\partial x}$$

$$-\frac{1}{\rho} \frac{\partial p}{\partial y} = U \frac{\partial u}{\partial x}$$

$$-\frac{1}{\rho} \frac{\partial p}{\partial z} = U \frac{\partial w}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots(21)$$

where the terms of second order magnitude have been neglected.

Under the same assumptions, the equation of continuity which, in full, is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{1}{a^2}\left(\frac{u}{\rho}\frac{\partial p}{\partial x} + \frac{v}{\rho}\frac{\partial p}{\partial y} + \frac{w}{\rho}\frac{\partial p}{\partial z}\right) = 0\ , \quad ..(22)$$

/becomes.....

becomes, taking into account (21),

$$-\beta^2 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

where $\beta^2 = M^2 - 1 = \frac{U^2}{a^2} - 1$, $M = \frac{U}{a}$ being the Mach number.

Equation (21) is the linearised equation of continuity. If, in addition, the flow is irrotational, then we have curl $\underline{q} = 0$, where $\underline{q} = (u, v, w)$, i.e.,

$$\frac{\partial w}{\partial y} - \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \ldots(24)$$

In that case, there exists a velocity potential so that $\underline{q} = -$ grad . Expressing u, v, w in terms of in (23), we obtain (20).

Equation (22) expresses the fact that div $\underline{q}' = 0$ where the 'current vector' $\underline{q}'$ is defined by $\underline{q}' = \rho\, q$. Since (23), which is the linearised version of (22), indicates that div$[\rho(-\beta^2 u, v, w)] = 0$, it will be seen that the corresponding 'linearised current vector' is $\underline{q}' = \rho(-\beta^2 u, v, w)$, where ρ is now constant. Dividing $\underline{q}'$ by ρ we obtain a vector $\underline{q}^* = (-\beta^2 u, v, w)$ which will be called the reduced current velocity, or short c-velocity of the flow. Thus, apart from the flow of $\underline{q}$ across a surface S, $\int \underline{q}\, d\underline{S}$ we are led to consider also the flow of $\underline{q}'$ across S. In order to distinguish between the two types of flow, the flow of $\underline{q}'$ will be called 'C - flow'. It will be seen that the linearised equation of continuity (23) is the differentia l expression of the fact that the total c-flow across a closed surface vanishes.

It now becomes natural to introduce alongside the conventional operators ∇, grad, div, and Δ, the operators $\nabla\underline{q}\beta$, gradhβ, divhβ, Δhβ. (Read 'hyperbolic nabla of index β', 'Hyperbolic gradient of index β, etc. The index β will normally be fixed throughout and may therefore be omitted.) The operators $\nabla h\beta$ will be defined by $\nabla h\beta = \left(\beta^2 \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, and gradh$\beta$ and divhβ as the two modes of this operator which apply to scalars, and to vectors in scalar multiplication respectively. The operator $\Delta h\beta$ is then defined by $\Delta h\beta =$ div gradhβ = divhβ grad. Equation (20) may now be written

$$\Delta h \Phi = \text{div gradh} \Phi = \text{divh grad} \Phi = 0 \ldots\ldots(25)$$

By the divergence theorem we have, for any function which is sufficiently regular on and inside a closed surface S bounding a volume V,

$$\int_S \text{gradh}\Phi\, d\underline{S} = \int_V \Delta h \Phi\, dV \quad \ldots\ldots\ldots\ldots\ldots(26).$$

/If, ...

If, in addition, Φ is a solution of (25) then

$$\int_S \operatorname{gradh} \Phi \, d\underline{S} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$$

However, if we replace ordinary integrals by finite parts, then equation (26) holds even when infinites are involved provided gradh Φ and $\Delta h \Phi$ are admissible functions with respect to a certain surface Σ. Thus, in that case

$$\int_S^{*} \operatorname{gradh} \Phi \, d\underline{S} = \int_V^{*} \Delta h \Phi \, dV \quad \ldots\ldots\ldots (28)$$

while (27) becomes

$$\int_S^{*} \operatorname{gradh} \Phi \, d\underline{S} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (29)$$

In the sequel, such surfaces Σ will be frequently composed of cones of the form $(x-x_o)^2 - \beta^2\left[(y-y_o)^2+(z-z_o)^2\right] = 0$

We notice that if a function ψ is admissible with respect to a surface Σ and another function Φ is regular (or has derivatives of sufficiently high order) on Σ, then the product $\psi\,\Phi$ is admissible. (The product of two admissible functions is not in general admissible). We also notice that if a finite number of functions are involved, admissible with respect to different surfaces $\Sigma^{(n)}$, then they will all be admissible with respect to one and the same surface, viz., the sum of the different $\Sigma^{(n)}$, given by the product of the functions $D^{(n)}$ defining the $\Sigma^{(n)}$.

Let Φ be a function which is regular (or has derivatives of a sufficiently high order) on and inside a volume V bounded by a surface S, and ψ a function which, together with its first and second derivatives is admissible in V (with respect to some algebraic surface Σ). Applying (9) to $f = \psi \operatorname{gradh} \Phi$, we obtain

$$\int_S^{*} \psi \operatorname{gradh} \Phi \, d\underline{S} = \int_V^{*} \operatorname{grad}\psi \operatorname{gradh}\Phi \, dV + \int_V^{*} \psi \Delta h \, \Phi \, dV \quad .(30)$$

and similarly

$$\int_S^{*} \Phi \operatorname{gradh}\psi \, d\underline{S} = \int_V^{*} \operatorname{grad}\Phi \operatorname{gradh}\psi \, dV + \int_V^{*} \Phi \Delta h \, \Phi \, dV \quad \ldots . (31)$$

/Now............

Now $\operatorname{grad}\psi \operatorname{gradh}\Phi = -\beta^2 \frac{\partial\psi}{\partial x}\frac{\partial\Phi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{\partial\Phi}{\partial y} + \frac{\partial\psi}{\partial z}\frac{\partial\Phi}{\partial z}$

$= \operatorname{gradh}\psi \operatorname{grad}\Phi$.

Hence, subtracting (31) from (30), we obtain

$$\overset{*}{\int}_S \left(\psi \operatorname{gradh}\Phi - \Phi \operatorname{gradh}\psi\right) d\underline{S} = \overset{*}{\int}_V \left(\psi\Delta h\Phi - \Phi\Delta h\psi\right) dV \quad \ldots\ldots(32)$$

This is the counterpart of Green's formula, extended however to include finite parts (compare refs. 1 and 3).

4. Source and doublet distributions in steady supersonic flow

Elementary solutions of equations (20) or (25) are the functions Φ_P (x, y, z) defined by

$$\left.\begin{aligned}\Phi_P(x,y,z) &= \frac{\sigma}{\sqrt{(x-x_0)^2 - \beta^2\left[(y-y_0)^2+(z-z_0)^2\right]}} \quad \text{for } (x-x_0)^2 > \beta^2\left[(y-y_0)^2\ (z-z_0)^2\right], x > x_0 \\ \text{and} \\ \Phi_P(x,y,z) &= 0 \quad \text{elsewhere}\end{aligned}\right\} (33)$$

where $P = (x_0, y_0, z_0)$ and σ are arbitrary. Φ_P will be said to be the velocity potential of a source of strength σ located (or, 'with origin') at P. The actual velocity potential of a (weak) source travelling at a velocity -U in a field of reference travelling with the source, is obtained by adding -Ux to Φ_P as given by (33). For reasons of simplicity, Φ_P as in (33) will be called a source for positive as well as for negative σ.

Similarly, a function ψ_P will be said to be the velocity potential of a counter-source of strength located at P if it is given by

$$\left.\begin{aligned}\psi_P(x,y,z) &= \frac{\sigma}{\sqrt{(x-x_0)^2 - \beta^2\left[(y-y_0)^2+(z-z_0)^2\right]}} \quad \text{for } (x-x_0)^2 > \beta^2\left[(y-y_0)^2+(z-z_0)^2\right], x < x_0 \\ \text{and} \\ \psi_P(x,y,z) &= 0 \quad \text{elsewhere}\end{aligned}\right\} (34)$$

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A 'doublet' is obtained by differentiating Φ_P with respect to length in any given direction, the differentiation being performed relative to the coordinates of P. Thus the velocity potential $\bar{\Phi}_P$ of a doublet whose 'axis' is parallel to the z-axis is given by

$$\left.\begin{aligned} \bar{\Phi}_P(x,y,z) &= \frac{-\sigma\beta^2(z-z_0)}{\left[(x-x_0)^2-\beta^2\left[(y-y_0)^2+(z-z_0)^2\right]\right]^{3/2}} \\ &\qquad \text{for } (x-x_0)^2 > \beta^2\left[(y-y_0)^2+(z-z_0)^2\right] \text{ and } x > x_0 \\ &\text{and} \\ \bar{\Phi}_P(x,y,z) &= 0 \qquad\qquad \text{elsewhere} \end{aligned}\right\} \quad (35)$$

A 'counter-doublet' is obtained by applying a similar operation to Ψ_P. An asterisk will be employed to indicate fundamental solutions of unit strength (e.g., Φ^*_P).

It will be seen that the velocity potentials of sources, counter-sources, doublets, etc., and all their derivatives are admissible functions in all regions excluding their origins, the surface Σ being given by

$$(x-x_0)^2 - \beta^2\left[(y-y_0)^2+(z-z_0)^2\right] = 0.$$

Also, the potentials of sources and counter-sources tend to 0 of order ½ as the affix tends to infinity in any direction not asymptotic to Σ, and similarly doublets and counter-doublets tend to 0 of order 3/2 under the same conditions.

The velocity potentials due to line, surface, and volume distributions in points outside the distributions are obtained by evaluating the integrals $\int\sigma\Phi^*_P\,d\lambda$, $\int\sigma\Phi^*_P\,dS$, $\int\sigma\Phi^*_P\,dV$, where σ denotes the (variable) line surface or volume density, and Φ^*_P denotes the velocity potential of a source of unit strength the coordinates of whose origin coincide with the variable(s) of integration. For sufficiently regular distribution functions σ, (e.g., if σ has continuous bounded first derivatives) these integrals exist as ordinary improper integrals. For instance, for a surface distribution we obtain

$$\Phi(x,y,z) = \int\frac{\sigma\,dS}{\sqrt{(x-x_0)^2+\beta^2\left[(y-y_1)^2+(z-z_0)^2\right]}} \quad \ldots\ldots\ldots(36)$$

where σ is defined as a function of the parameters u, v, of the surface S, given by $x_0 = y_0(u,v)$, $z_0 = z_0(u,v)$ and

the integral is taken over those parts of the surface for which $(x-x_0)^2 > \beta^2\left[(y-y_0)^2+(z-z_0)^2\right]$ and $x > x_0$.

/The.....

The position is different in the case of line, surface, or volume distributions of doublets, since the integrals corresponding to such distributions, viz., $\int \sigma \Phi^{*}_{P}\, d\ell$, $\int \sigma \Phi^{*}_{P}\, dS$, and $\int \sigma \Phi^{*}_{P}\, dV$ are in general infinite. Thus, for a surface distribution of doublets whose axes are all parallel to the z-axis we obtain

$$\int \frac{-\sigma(z-z_0)\beta^2\, dS}{\left[(x-x_0)^2 - \beta^2\left[(y-y_0)^2 + (z-z_0)^2\right]\right]^{3/2}} \quad \ldots\ldots.(37)$$

which is, in general, infinite. However, provided σ is sufficiently regular, the finite part of the integral still exists, and we may say that

$$\overset{*}{\Phi}\!\int \sigma\, \Phi^{*}_{P}\, dS(\mathrm{curl}\,\ell)$$

is the potential due to a doublet distribution over S, (with similar definitions for potentials due to line or volume distribution.. An alternative method which avoids the use of the finite part and which has been used by Schlickting (ref.2) and others, is to consider first the corresponding integral for sources (eqn.(36)) and then to differentiate with respect to $(-z)$. From a physical point of view this means that we calculate the potential due to two infinitely near source distributions of opposite strength. The final result is the same since, according to the rules given in the preceding paragraph, finite parts can always be differentiated under the sign of the integral. It is precisely the possibility of carrying out all the necessary operations directly, without fear (or certainty) of obtaining meaningless symbols,which makes the finite part such a convenient concept. It will be seen that the alternative method is applicable only when all the doublets have parallel axes.

In order to define the potential due to a volume distribution of sources (or of counter-sources) we have, as in classical theory, to take recourse to a limit process, as the integrand $\sigma \Phi_{P}$ tends to ∞ of the order I on approaching the point for which the potential is calculated. We therefore surround the point by a small sphere of radius ϵ, evaluate the integral excluding the interior of the sphere, and then let ϵ tend to 0. For finite ϵ, the integrals in question exist as ordinary improper integrals, and the limit exists, as ϵ tends to 0, since the volume of the sphere tends to 0 as ϵ^3.

We are now going to show that the c-flow - defined as a finite part - across a closed surface surrounding a source of strength σ is equal to $2\pi\sigma$.

/We.....

We have to prove

$$^{*}J = -\, ^{*}\!\!\int_S \operatorname{gradh} \bar{\Phi}_P \, d\underline{S} = 2\pi\sigma \quad \ldots\ldots\ldots\ldots(38)$$

where $\bar{\Phi}_P$ is defined by (33).

We may simplify the problem without loss of generality by assuming $x_o = y_o = z_o = 0$. Now let S be a small cylindrical surface bounded by two planes $x = \pm\alpha$ and by the cylinder $y^2 + z^2 = r^2$ where $r > \frac{\alpha}{\beta}$. Then the integral of (35) vanishes everywhere on S except in the circular area belonging to the plane $x = \alpha$. Hence $^{*}J$ reduces to

$$^{*}J = \, ^{*}\!\!\int_{y^2 + z^2 < r^2} \frac{\sigma\alpha\beta^2 dydz}{\left[\alpha^2 - \beta^2(y^2 + z^2)\right]^{3/2}}$$

and therefore $^{*}J = 2\pi\sigma$, by (8). This confirms the theorem for the particular case of a circular cylinder.

Next, let S be an arbitrary surface surrounding the source, then we may find a small cylindrical surface S' of the above description inside S and we only have to show that $^{*}\!\!\int_S \operatorname{gradh} \bar{\Phi}_P \, d\underline{S} = \, ^{*}\!\!\int_{S'} \operatorname{gradh} \bar{\Phi}_P d\underline{S}$.
Let V be the volume bounded by S and S'. Then by the divergence theorem for finite parts, (9),

$$^{*}\!\!\int_S \operatorname{gradh} \bar{\Phi}_P \, d\underline{S} - \, ^{*}\!\!\int_{S'} \operatorname{gradh} \bar{\Phi} \, d\underline{S} = \, ^{*}\!\!\int_V \operatorname{div} \operatorname{gradh} \bar{\Phi}_P dV$$

But

$$^{*}\!\!\int_V \operatorname{div} \operatorname{gradh} \bar{\Phi}_P \, dV = \, ^{*}\!\!\int_V \Delta h \bar{\Phi}_P \, dV = 0,$$ since $\bar{\Phi}$ satisfies (25) and V does not include the origin. This proves that (38) is true generally.

Similarly, we obtain for counter sources whose potential Ψ_P is given by (34).

$$-\, ^{*}\!\!\int_S \operatorname{gradh} \Psi_P \, dS = 2\pi\sigma \quad \ldots\ldots\ldots\ldots\ldots(39)$$

/More.......

More generally, if a surface S surrounds a finite number of sources of strengths σ_n superimposed on an arbitrary field of flow which is regular inside S, then

$$^{*}\!\!\int_S \operatorname{gradh} \Phi \, d\underline{S} = 2\pi \sum \sigma_n \qquad \ldots\ldots\ldots(40)$$

There is a similar theorem for counter sources.

In fact, (40) follows immediately from (29) and (38).

We may also deduce, taking into account (i), at the end of para. 2, that the c-flux across a surface surrounding a doublet vanishes, and more generally that the flux across a closed surface is not affected by the superposition of doublets either inside it or outside.

Finally, it follows from (ii) at the end of para.2, that (38) can also be applied to a continuous distribution of sources inside a surface S, so

$$- ^{*}\!\!\int_S \operatorname{gradh} \Phi \, d\underline{S} = 2\pi \int \sigma \qquad \ldots\ldots\ldots\ldots\ldots\ldots(41)$$

where $\int \sigma$ is the total strength of the sources enclosed by S (and given as a line, surface, or volume distribution). The same applies in the limit when the distribution is actually bounded in parts by S.

Equation (41) shows that the finite part of an infinite integral is more than an artificial analytical concept, and that in certain cases it may have a definite physical meaning. In fact, the product of density and c-flux, $-\rho \int \operatorname{gradh} \Phi \, d\underline{S}$ is the linearised expression for the total flux of matter whenever that expression exists, for instance if Φ is given by a homogeneous volume distribution of sources over the interior of a small sphere of radius and centre P inside S, together with $-Ux$ corresponding to the free stream velocity. If σ is the total strength of the distribution, then according to (41), $-\rho \int \operatorname{gradh} \Phi \, d\underline{S} = 2\pi\sigma\rho$. Thus, $2\pi\sigma\rho$ is the rate at which matter is produced inside S. Now let ϵ tend to 0, while σ is kept constant. Then Φ tends to $-Ux + \Phi_P$ where Φ_P is the potential of a source of strength σ located at P. But σ having been kept constant, it follows that the rate at which matter is produced inside S, and hence, the rate at which matter crosses S is still $2\pi\sigma\rho$. And this, by (38), can be expressed by

$$-\rho \, ^{*}\!\!\int \operatorname{gradh} \Phi_P \, d\underline{S} = -\rho \, ^{*}\!\!\int \operatorname{gradh} \left(-Ux + \Phi_P\right) d\underline{S},$$

so that the finite part is the natural generalisation of an ordinary integral when the latter diverges.

/Applying....

Applying (41) to a small surface surrounding a point P inside a volume distribution of sources, we obtain

$$- \int^{*}_{S} \mathrm{gradh}\, \Phi \, d\underline{S} = 2\pi \int_{V} \sigma \, dV$$

and transforming the left hand side by means of the divergence theorem this becomes

$$- \int^{*}_{V} \mathrm{div}\ \mathrm{gradh}\, \Phi \, dV = 2\pi \int_{V} \sigma \, dV$$

Since this is true for an arbitrary small volume containing P, we must have

$$\Delta h \Phi = \mathrm{div}\ \mathrm{gradh}\ \Phi = - 2\pi\sigma \ \ldots\ldots\ldots\ldots (42)$$

which is the counterpart of Poisson's theorem in subsonic theory.

Conversely, given the differential equation (42) over a certain region R, a particular solution of it is

$$\Phi = \int_{R} \sigma \Phi^{*}{}_{P}\, dV \qquad \ldots\ldots\ldots\ldots\ldots\ldots (43)$$

The general solution, as is easily seen by subtraction, then is $\Phi = \int_{R} \sigma \Phi^{*}{}_{P}\, dV + \Phi$ where Φ is an arbitrary solution of (25).

Given a surface distribution of sources, it can be shown that the components of the gradient and therefore of the hyperbolic gradient of the potential remain finite and continuous on either side of the surface S. Also, Φ, and therefore its tangential derivatives, are continuous across the surface.

In order to find the discontinuity of the normal derivative across S, we apply (41) to a small cylinder whose bases are parallel to the surface on either side of it, and whose height is again small compared with its lateral dimensions (Fig.2). Letting first the height of the cylinder tends to 0 we find that $\int_{S'_+} \mathrm{gradh}\, \Phi \, d\underline{S} - \int_{S'_-} \mathrm{gradh}\, \Phi \, d\underline{S} = 2\pi \int_{S'} \sigma \, dS$ where S' denotes the portion of S inside the cylinder, and S'_+ and S'_- denote the two bases of the cylinder respectively, S'_+ being the base whose outside normal coincides with the outside normal of S. Letting S tend to 0 round any given point on S, we obtain, denoting by λ, μ, ν the direction cosines of S in the point in question, and indicating by $\pm$ the derivatives of Φ on the two sides respectively,

/(44)....

$$-\lambda\beta^2\left(\frac{\partial\Phi}{\partial x_+}-\frac{\partial\Phi}{\partial x-}\right)+\mu\left(\frac{\partial\Phi}{\partial y_+}-\frac{\partial\Phi}{\partial y-}\right)+\nu\left(\frac{\partial\Phi}{\partial z_+}-\frac{\partial\Phi}{\partial z-}\right)=2\pi\sigma \quad .....(44)$$

In particular, if the distribution is in the $x - y$ plane, we have $\lambda=\mu=0, \nu=1$, so that

$$\frac{\partial\Phi}{\partial z_+}-\frac{\partial\Phi}{\partial z-}=2\pi\sigma \quad ....(45)$$

a result which is of considerable importance for the calculation of the wave drag of an aerofoil moving at supersonic speed at zero incidence (refs. 4, 5, 6). A similar relation for the discontinuity of the potential across a doublet distribution in the $x\,y$ plane, and which can be derived from (42), is fundamental in the supersonic theory of flat aerofoils at incidence.

These relations have hitherto been inferred by analogy with incompressible theory and then proved ad hoc in the particular cases required. The need for a more systematic development was pointed out in the introduction to ref. 5.

Dividing (44) by $\Delta=\sqrt{(\lambda\beta)^2+\mu^2+\nu^2}$ we obtain the result that the discontinuity of $\frac{\partial\Phi}{\partial n'}$ in a direction $\underline{n}'$ whose components (direction cosines) are $-\frac{\lambda\beta^2}{\Delta}, \frac{\mu}{\Delta}, \frac{\nu}{\Delta}$ is $\frac{2\pi\sigma}{\Delta}$. And since the tangential derivatives of Φ are continuous across the surface, it follows that the discontinuity of $\frac{\partial\Phi}{\partial n'}$ must be the discontinuity of $\frac{\partial\Phi}{\partial n}$, where $\underline{n}=(\lambda, \mu, \nu)$ is normal to S, multiplied by the cosine between $\underline{n}$ and $\underline{n}'$. Hence

$$\frac{\partial\Phi}{\partial n_+}-\frac{\partial\Phi}{\partial n_-}=2\pi\sigma\left(-\lambda^2\beta^2+\mu^2+\nu^2\right)\big/\Delta \quad \text{or}$$

$$\frac{\partial\Phi}{\partial n_+}\;\frac{\partial\Phi}{\partial n_-}=2\pi\frac{1-\lambda^2(1+\beta^2}{\sqrt{1-\lambda^2(1-\beta^4)}}\sigma \quad ....(46)$$

Equation (46) amends the statement in ref. 5 that the discontinuity of the normal derivatives is always $2\pi\sigma$. The particular case in which this statement was applied, however, viz. (45), remains correct.

The chief use to which Hadamard puts his concept of the finite part is related to the above applications but is the outcome of a rather different approach. Hadamard's purpose is the solution of Cauchy's initial value problem for a very general class of hyperbolic partial differential equations including (20) as a special case.

/We.....

We are going to develop Hadamard's result in respect of equation (20), i.e. we are going to find an expression for the value of a solution Φ of (20) in a point $P = (x_0, y_0, z_0)$ inside a closed surface S, when the values of Φ and of $\frac{\partial \Phi}{\partial n'}$ are known on S, where for every point of S, the direction $\underline{n}'$ is defined as above. For this particular case, an equivalent formula had been derived previously by Volterra.

Let ψ^*_P be the velocity potential of a counter source of unit strength located at P; then the function $\Phi \psi^*_P$ is admissible inside S, excluding only P, provided Φ is regular inside S - and this can in fact be verified a posteriori.

Let us surround P by a small surface S' and apply equation (32) to the volume V bounded by S + S'. Since $\Delta h \Phi = \Delta h \psi^*_P = 0$, we obtain

$$^*\int_{S+S'} (\Phi \operatorname{gradh} \psi^*_P - \psi^*_P \operatorname{gradh} \Phi) d\underline{S} = 0$$

or, taking the <u>inward</u> normal as the direction of a surface element of S, and the outward normal as the direction of a surface element of S'

$$^*\int_{S'} (\Phi \operatorname{gradh} \psi^*_P - \psi^*_P \operatorname{gradh} \Phi) d\underline{S} = {}^*\int_{S} (\Phi \operatorname{gradh} \psi^*_P - \psi^*_P \operatorname{gradh} \Phi) d\underline{S} \quad \ldots\ldots(47)$$

It can be shown that as S' contracts to the point P, the left hand side of (47) tends to $2\pi\Phi(x_0, y_0, z_0)$. Thus

$$2\pi\Phi(x_0, y_0, z_0) = {}^*\int_{S} (\Phi \operatorname{gradh} \psi^*_P - \psi^*_P \operatorname{gradh} \Phi) d\underline{S} \quad \ldots..(48)$$

Denoting by λ, μ, ν the direction cosines of the normal to dS, we have, for any scalar function Φ,

$$\operatorname{gradh} \Phi \, d\underline{S} = (-\lambda\beta^2 \frac{\partial \Phi}{\partial x} + \mu \frac{\partial \Phi}{\partial y} + \nu \frac{\partial \Phi}{\partial z}) dS$$

For $\beta = 1$, this becomes $\operatorname{gradh} \Phi \, d\underline{S} = - \frac{\partial \Phi}{\partial n^*} dS$, where $\underline{n}^* = (\lambda, -\mu, -\nu)$. Hence, for $\beta = 1$

$$2\pi\Phi(x_0, y_0, z_0) = {}^*\int_{S} (\psi^*_P \frac{\partial \Phi}{\partial n^*} - \Phi \frac{\partial \psi}{\partial n^*}) dS \quad \ldots\ldots\ldots(49)$$

This is Hadamard's formula (58) (ref. 1, p.207) for the special case f = 0. The direction $\underline{n}^*$ is called by Hadamard the transversal direction to dS. Its geometrical interpretation, due to Coulon is that it is conjugate to the tangent plane to dS with respect to the cone $(x-x_1)^2 - \left[(y-y_1)^2+(z-z_1)^2\right] = 0$, whose vortex is located at dS.

6. Vorticity distributions in steady supersonic flow.

We now direct our attention to the study of rotational motion.

Given a field vector $\underline{q} = (u, v, w)$ we denote by ξ, η, ζ the components of curl $\underline{q}$, $\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$, $\eta = \frac{\partial u}{\partial z} - \frac{\partial v}{\partial x}$, $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. The differential equation of the system of vortex lines is $\frac{dx}{\xi} \quad \frac{dy}{\eta} \quad \frac{dz}{\zeta}$ as usual, and the strength of a vortex tube is defined as the product of the cross section σ into the resultant vorticity $w = \left(\xi^2 + \eta^2 + \zeta^2\right)^{\frac{1}{2}}$ and is the same at all points of a vortex. All these results and definitions are in fact quite independent of whether the fluid is compressible or incompressible, except that in the case of supersonic flow, it may be necessary to consider the finite parts of integrals of the type $\int_C \underline{q}\ d\underline{\ell}$ and $\int_S$ curl $\underline{q}\ d\underline{S}$ in cases where the ordinary integrals do not exist.

Applying the vector operator ∇h in cross multiplication to a vector $\underline{q} = (u, v, w)$, we obtain a vector which will be called curlh $\underline{q}$ (hyperbolic curl of $\underline{q}$).

Explicitly -

$$\text{curlh } \underline{q} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ \frac{\partial u}{\partial z} + \beta^2 \frac{\partial w}{\partial x},\ -\beta^2\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (50)$$

Direct calculation shows that

$$\text{divh curlh } \underline{q} = 0 \qquad (51)$$

and

$$\text{curl curlh } \underline{q} = \text{gradh div } \underline{q} - \text{div gradh } \underline{q} \qquad (52)$$

A field vector $\underline{q}$ will be called irrotational or lamellar, as usual, if curl $\underline{q} = 0$, and it will be called hyperbolic solenoidal if divh $\underline{q} = 0$.

We are going to show that a vector $\underline{q}$ defined in a region R and admissible in it can be represented as the sum of three vectors, one irrotational, one hyperbolic solenoidal, and one both irrotational and hyperbolic solenoidal.

More precisely, we are going to represent $\underline{q}$ as $\underline{q} = \underline{q}_1 + \underline{q}_2 + \underline{q}_3$ where

$$\text{divh } \underline{q}_1 = \text{divh } \underline{q}, \quad \text{curl } \underline{q}_1 = 0 \quad \text{in } R \qquad (53)$$

/(54)..

$$\operatorname{curl} \underline{q}_2 = \operatorname{curl} \underline{q} \quad , \quad \operatorname{divh} \underline{q}_2 = 0 \quad \text{in R} \quad \ldots.(54)$$

and

$$\operatorname{divh} \underline{q}_3 = 0 \ , \ \operatorname{curl} \underline{q}_3 = 0 \qquad \text{in R} \quad \ldots.(55)$$

Assuming that vectors as described in (53) and (54) have been found, we put $\underline{q}_3 = \underline{q} - \underline{q}_1 - \underline{q}_2$. Then $\operatorname{divh} \underline{q}_3 = \operatorname{divh} \underline{q} - \operatorname{divh} \underline{q}_1 - \operatorname{divh} q_2 = 0$, and $\operatorname{curl} \underline{q}_3 = \operatorname{curl} \underline{q} - \operatorname{curl} \underline{q}_1 - \operatorname{curl} \underline{q}_2 = 0$, so that $\underline{q}_3$ defined in that way satisfies (55).

Putting $\sigma = \frac{1}{2\pi} \operatorname{divh} \underline{q}$ in R, we determine a scalar function Φ by $\Phi = \overset{*}{\int_R} \sigma \overset{*}{\Phi}_P \, dV$, so that $\Delta h \Phi = -2\pi\sigma$ according to (42), i.e., $\Delta h \Phi = -\operatorname{divh} \underline{q}$, and so $\underline{q}_1 = -\operatorname{grad} \Phi$ satisfies (53). Thus

$$q_1 = -\frac{1}{2\pi} \operatorname{grad} \overset{*}{\int_R} \operatorname{divh} \underline{q} \cdot \overset{*}{\bar{\Phi}}_P \, dV \quad ..(56)$$

To find $\underline{q}_2$, we shall assume that $\underline{q}_2$ is given as the hyperbolic curl of a vector $\underline{\underline{\Psi}} = (\psi^1, \psi^2, \psi^3)$, $\underline{q}_2 = \operatorname{curlh} \underline{\underline{\Psi}}$, and so (57) $\operatorname{curl} \underline{q}_2 = \operatorname{curl} \operatorname{curlh} \underline{\underline{\Psi}} = \operatorname{gradh} \operatorname{div} \underline{\underline{\Psi}} - \operatorname{div} \operatorname{gradh} \underline{\underline{\Psi}}$, by (52). We now restrict $\underline{\underline{\Psi}}$ by the condition that $\operatorname{div} \Psi = 0$. Then we must have $\operatorname{div} \operatorname{gradh} \underline{\underline{\Psi}} = -\operatorname{curl} \underline{q}_2 = -\operatorname{curl} \underline{q}$, by (54) and (57)., i.e.,

$$\Delta h \psi^1 = -\xi \ , \ \Delta h \psi^2 = -\eta \ , \ \Delta h \psi^3 = -\zeta \ \text{in R} \ ..(58)$$

and this according to (42) is solved by

$$\psi^1 = \frac{1}{2\pi} \overset{*}{\int_R} \xi \overset{*}{\bar{\Phi}}_P \, dV, \quad \psi^2 = \frac{1}{2\pi} \overset{*}{\int_R} \eta \overset{*}{\bar{\Phi}}_P \, dV \ ,$$

$$\psi^3 = \frac{1}{2\pi} \overset{*}{\int_R} \zeta \overset{*}{\bar{\Phi}}_P \, dV$$

or

$$\underline{\Psi} = \frac{1}{2\pi} \overset{*}{\int_R} \operatorname{curl} \underline{q} \, \overset{*}{\bar{\Phi}}_P \, dV \qquad \ldots\ldots\ldots\ldots(59)$$

/Then......

Then $\underline{q}_2 = \text{curlh}\,\underline{\underline{\Psi}}$ satisfies (54), provided we can show that in fact $\text{div}\,\underline{\underline{\Psi}} = 0$, as assumed. And this can be shown exactly as in the classical counterpart (ref. 7, para.148), provided the integrand vanishes at infinity or, in particular, if it vanishes outside a finite region. This again will certainly be the case if curl $\underline{q}$ vanishes for sufficiently small x, since Φ_P vanishes for sufficiently large x.

It is clear from the above construction that $\underline{q}_1$ determines the flow due to the source distribution in R, while $\underline{q}_2$ represents the flow due to the vorticity distribution. Given a three dimensional vorticity distribution we then have in detail

$$\underline{\underline{\Psi}}(x,y,z) = \frac{1}{2\pi} \overset{*}{\int}_{R'} (\xi, \eta, \zeta) \frac{dx_o\, dy_o\, dz_o}{\sqrt{(x-x_o)^2 - \beta^2[(y-y_o)^2+(z-z_o)^2]}} \quad \text{............(60)}$$

(The expression on the right hand side is actually an ordinary improper integral), where R' is the sub-domain of R which satisfies $(x-x_o)^2 > \beta^2[(y-y_o)^2 + (z-z_o)^2]$ and $x_o < x$.

We then obtain for the components u, v, w, of $\underline{q}_2 = \text{curlh}\,\underline{\underline{\Psi}}$,

$$u = \frac{\beta^2}{2\pi} \overset{*}{\int}_{R'} [(z-z_o)\eta - (y-y_o)\zeta] \frac{dx_o\, dy_o\, dz_o}{[(x-x_o)^2 - \beta^2[(y-y_o)^2+(z-z_o)^2]]^{3/2}}$$

$$v = \frac{\beta^2}{2\pi} \overset{*}{\int}_{R'} [(x-x_o)\zeta - (z-z_o)\xi] \frac{dx_o\, dy_o\, dz_o}{[(x-x_o)^2 - \beta^2[(y-y_o)^2+(z-z_o)^2]]^{3/2}}$$

$$w = \frac{\beta^2}{2\pi} \overset{*}{\int}_{R'} [(y-y_o)\xi - (x-x_o)\eta] \frac{dx_o\, dy_o\, dz_o}{[(x-x_o)^2 - \beta^2[(y-y_o)^2+(z-z_o)^2]]^{3/2}} \quad \text{................(61)}$$

This may be written

$$\underline{q}_2 = - \overset{*}{\int}_{R'} (\underline{r} \times \text{curl}\,\underline{q}) \frac{dV}{s^3} \quad \text{....................(62)}$$

where $\underline{r} = (x-x_o,\ y-y_o,\ z-z_o)$, and $s = [(x-x_o)^2 - \beta^2[(y-y_o)^2 + (z-z_o)^2]]^{\frac{1}{2}}$

The corresponding formula for incompressible flow is

$$\underline{q}_2' = \overset{*}{\int}_{R'} (\underline{r} \times \text{curl}\,\underline{q}') \frac{dV}{r^3} \quad \text{.........(63)}$$

/The.....

The discrepancy in sign is only apparent, since as, for instance, formula (8) shows, the sign of a finite part does not follow the sign of the integrand, as for ordinary integrals.

We may now calculate the field of flow due to an isolated re-entrant line vortex C. Replacing the volume element $dx_o\,dy_o\,dz_o$ in (60) by $\sigma_o d\ell_o$ where $d\ell_o$ is the element of length of C, and σ_o its infinitesimal cross section, and writing $\omega = (\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}}$, we have

$\xi = \omega\dfrac{dx_o}{d\ell_o}$, $\eta = \omega\dfrac{dy_o}{d\ell_o}$, $\zeta = \omega\dfrac{dz_o}{d\ell_o}$, and so, since $\omega\sigma_o$ is a constant K, and since $\underline{d\ell_o} = (dx_o, dy_o, dz_o)$, (60) becomes

$$\underline{\underline{\psi}}(x,y,z) = \frac{K}{2\pi}\int^{*}_{B'}\frac{\underline{d\ell_o}}{\sqrt{(x-x_o)^2 - \beta[(y-y_o)^2+(z-z_o)^2]}} \quad ..(64)$$

where C′ consists of the segments of C which satisfy $(x-x_o)^2 \geq \beta^2\left[(y-y_o)^2+(z-z_o)^2\right]$ and $x > x_o$.

If in particular C consists of straight segments which are parallel either to the x-axis or to the y-axis, then (64) can be integrated, since

$$\int\frac{dx_o}{\sqrt{(x-x_o)^2 - \beta^2[(y-y_o)^2+(z-z_o)^2]}} = -\cosh^{-1}\frac{x-x_o}{\beta\sqrt{(y-y_o)^2+(z-z_o)^2}} + \text{const.}$$

and

$$\int\frac{dy_o}{\sqrt{(x-x_o)^2 \cdot \beta^2[(y-y_o)^2+(z-z_o)^2}} = -\frac{1}{\beta}\sin^{-1}\frac{\beta(y-y_o)}{\sqrt{(x-x_o)^2 - \beta^2(z-z_o)^2}} + \text{const.}$$

$$\ldots\ldots\ldots\ldots(65)$$

Now let C be a 'horseshoe vortex' of strength K, ~~consisting~~ of the straight segments $(x_1 \leq x_o < \infty,\ y_o = -y_1,\ z_o = 0)$, $(x_o = x_1,\ -y_1 \leq y_o\ y_1,\ z_o = 0)$ and $(x_1 \leq x_o < \infty,\ y_o = y_1,\ z_o = 0)$ where x_1 and y_1 are given constants (Fig.2). Using (65), we find that $\psi^3 = 0$ always, and $\psi^1 = \psi^2 = 0$ for $x < x_1$, while for $x > x_1$,

/(66).......

$$\psi^1 = \frac{K}{2\pi}\left[\cosh^{-1}\frac{x - x_1}{\beta\sqrt{(y-y_1)^2 + z^2}} - \cosh^{-1}\frac{x - x_1}{\beta\sqrt{(y+y_1)^2 + z^2}}\right] \qquad \text{.........(66)}$$

where the $\cosh^{-1}$ are to be replaced by 0 when their arguments are smaller than 1, respectively, and

$$\psi^2 = -\frac{K}{2\pi\beta}\left[\sin^{-1}\frac{\beta(y-y_1)}{\sqrt{(x-x_1)^2 - \beta^2 z^2}} + \sin^{-1}\frac{\beta(y+y_1)}{\sqrt{(x-x_1)^2 - \beta^2 z^2}}\right] \qquad \text{..........(67)}$$

where the $\sin^{-1}$ are replaced $+\frac{\pi}{2}$ or $-\frac{\pi}{2}$ when their arguments are greater than 1, or smaller than -1, respectively.

We now obtain $\underline{q} = (u,v,w)$ by taking the hyperbolic curl of $\underline{\underline{\psi}} = (\psi^1, \psi^2, 0)$, so that $u = v = w = 0$ for $x_1 > x_1$ while for $x > x_1$,

$$u = \frac{K\beta^2}{2\pi}\left[\frac{(y - y_1)^2}{\left[(x-x_1)^2 - \beta^2 z^2\right]\sqrt{(x-x_1)^2 - \beta^2 (y-y_1)^2 + z^2}} - \frac{(y+y_1) z}{\left[(x-x_1)^2 - \beta^2 z^2\right]\sqrt{(x-x_1)^2 - \beta^2\left[(y+y_1)^2 + z^2\right]}}\right.$$

$$v = \frac{K}{2\pi}\left[\frac{(x-x_1) z}{\left[(y-y_1)^2 + z^2\right]\sqrt{(x-x_1)^2 - \beta^2\left[(y-y_1)^2 + z^2\right]}} - \frac{(x-x_1) z}{\left[(y+y_1)^2 + z^2\right]\sqrt{(x-x_1)^2 - \beta^2\left[(y+y_1)^2 + z^2\right]}}\right.$$

$$w = \frac{K}{2\pi}\left[\frac{(x-x_1)(y-y_1)\left[(x-x_1)^2 - \beta^2 (y-y_1)^2 + 2z^2\right]}{\left[(x-x_1)^2 - \beta^2 z^2\right]\left[(y-y_1)^2 + z^2\right]\sqrt{(x-x_1)^2 - \beta^2 (y-y_1)^2 + z^2}} - \frac{(x-x_1)(y+y_1)\left[(x-x_1)^2 - \beta^2\left[(y+y_1)^2 + 2z^2\right]\right]}{\left[(x-x_1)^2 - \beta^2 z^2\right]\left[(y+y_1)^2 + z^2\right]\sqrt{(x-x_1)^2 - \beta^2\left[(y+y_1)^2 + z^2\right]}}\right] \qquad \text{................(68)}$$

/where ...

where, for given x, y, z, the imaginary terms are omitted. Except for the notation, equations (68) agree with the field of flow round a horsewhoe vortex calculated by Schlichting by an entirelydifferent method (ref.2)

Some care is required when attempting to represent a volume or surface distribution of vortices as a combination of line vortices. Thus, according to (68), the components u, v, w, all vanish when (x, y,z) is outside both the cones $(x-x_1)^2 - \beta^2\left[(y \pm y_1)^2 + z^2\right]$ emanating from the tips

But it can be shown that this is no longer the case if the vorticity in the spanwise segment is distributed over a finite width Δx_0. However, even then the failure (which is due to the discontinuity of ψ for line vortices) can occur only at points belonging to the envelope of the cones of type

$$(x - x_0)^2 - \beta^2\left[(y-y_0)^2 + (z - z_0)^2\right] = 0$$

emanating from the vortex lines which are supposed to generate the surface or volume.

Applications to aerofoil theory are given in a separate report (ref.8).

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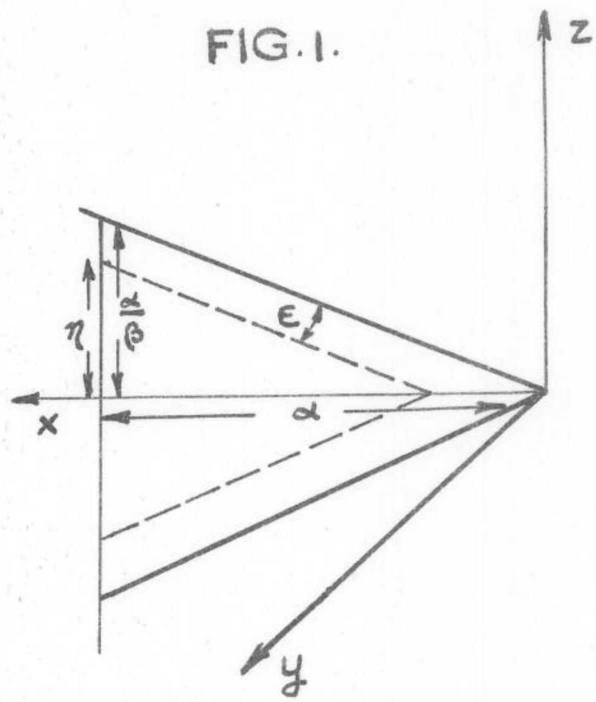
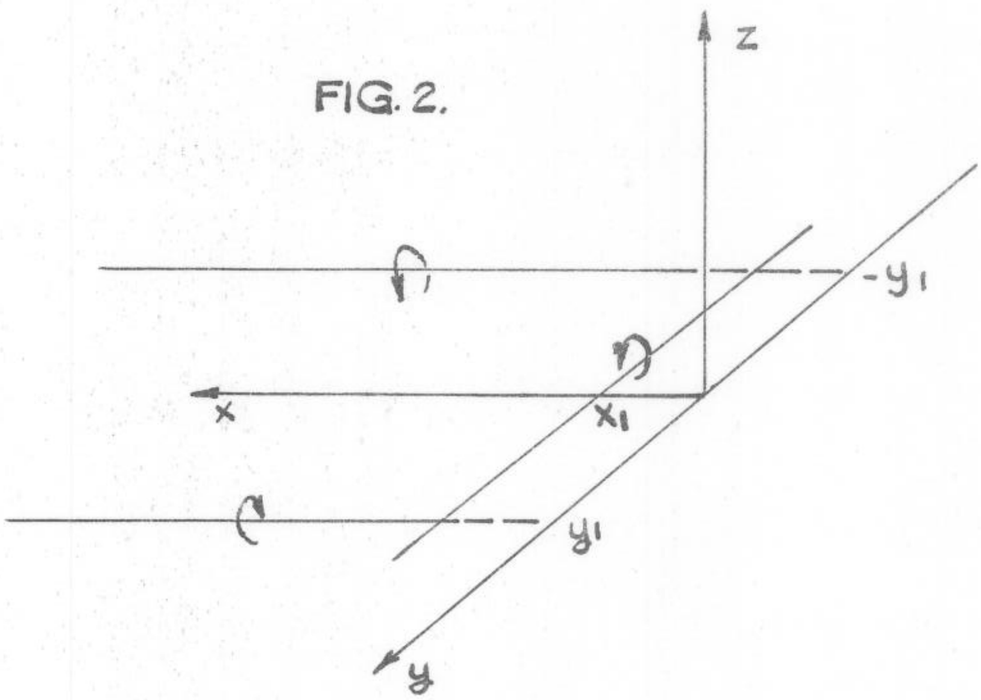
FIG. 1.



CALCULATION OF A FINITE PART.

FIG. 2.



CALCULATION OF THE FIELD OF FLOW
OF A SUPERSONIC HORSESHOE VORTEX.